Elliptic solutions to a generalized BBM equation

J. Nickel [a]

[a] *Department of Physics, University of Osnabrück, Barbarastr. 7, D-49069 Osnabrück, Germany*

Abstract

An approach is proposed to obtain some exact explicit solutions in terms of the Weierstrass' elliptic function $\wp$ to a generalized Benjamin-Bona-Mahony (BBM) equation. Conditions for periodic and solitary wave like solutions can be expressed compactly in terms of the invariants of $\wp$. The approach unifies recently established ad-hoc methods to a certain extent. Evaluation of a balancing principle simplifies the application of this approach.

Key words: BBM equation, elliptic solutions, solitons, Weierstrass' function
PACS: 02.30.Jr, 02.30.Gp

1 Introduction

The Benjamin-Bona-Mahony equation has been investigated as a regularized version of the Korteweg-de Vries equation for shallow water waves [1]. It incorporates nonlinear dispersive and dissipative effects [2,3]. In certain theoretical investigations the equation is superior as a model for long waves, and the word "regularized" refers to the fact that, from the standpoint of existence and stability, the equation offers considerable technical advantages over the Korteweg-de Vries equation. In addition to shallow water waves, the equation is applicable to the study of drift waves in plasma or the Rossby waves in rotating fluids. Under certain conditions, it also provides a model of one-dimensional transmitted waves. These find applications in semiconductor devices, optical devices, etc. [1]. - Apart from these applications solutions to the BBM equation are interesting in and of themselves.
Recently, Wazwaz [4] has introduced five new ansaetze, sinh-cosh-ansatz I - III, a sine-cosine- ansatz and a tanh-sech-method to find exact solutions to a

generalized BBM equation

$$u_t + u_x + a\, u^n u_x + u_{xxx} = 0,\ n \geq 1, \tag{1}$$

with constant parameter a. The ansaetze are direct approaches and "ad hoc" as a method, but yield (according to [4]) new results. Therefore, it seems reasonable to introduce and apply a further direct method to the above equation. As an interesting "feature" this method has compact conditions for periodic and solitary solutions. The power of the method is its ease of use and the existence of the mentioned conditions. Furthermore, it will be shown that the sinh-cosh-ansatz I-II, the sine-cosine-ansatz, the tanh-sech-method and the sinh-cosh-ansatz III for $n = 1$ are subcases of the method presented here.

The main features of this method are outlined in sec. 2 and applied to the BBM equation in brief. A balancing principle is evaluated in subsec. 2.1. Periodic and solitary solutions to the BBM are presented in subsec. 2.2. The relationship between this method and the ansaetze introduced in [4] are shown in sec. 3. A summary is given in sec. 4.

2 Outline of the method [5,6]

The starting point is a nonlinear wave and evolution equation (NLWEE)

$$\tau[u(\mathbf{x}), u'(\mathbf{x}), u''(\mathbf{x}), ..., u^{(k)}(\mathbf{x}), ..., u^{(m)}(\mathbf{x})] = t(\psi), \tag{2}$$

where τ is a function of $u(\mathbf{x})$ and its partial derivatives, the independent variable $\mathbf{x}$ has n components x_j, $u^{(k)}(\mathbf{x})$ denotes the collection of mixed derivative terms of order k and $t(u)$ is a trigonometric function in ψ or $t(u) = 0$. For notational simplicity the independent variables x, t will be used in the following. Equation (2) describes a certain dynamical system by means of a (wave) function $u(\mathbf{x})$. An travelling wave ansatz

$$u(x,t) \to g(f(z)) = \sum_{i=0}^{M} a_i f(z)^i,\ M \in \mathbb{N}, \tag{3}$$

where f is supposed to obey the nonlinear differential equation

$$\left(\frac{df(z)}{dz}\right)^2 = \alpha f^4 + 4\beta f^3 + 6\gamma f^2 + 4\delta f + \epsilon \equiv R(f), \tag{4}$$

(with real α, β, γ, δ, ϵ, $z = \mu(x - ct)$, $f(z)$) transforms the NLWEE into an polynomial equation $P(f) = 0$. Vanishing coefficients in the polynomial equation $P(f) = 0$ lead to equations which partly determine the coefficients α, β, γ, δ, ϵ in eq. (4). In general, the coefficients depend on the structure and parameters of the NLWEE and, finally, on the parameters of the transformation $\psi \to g$. The parameter M in eq. (3) can be derived by balancing (cf. subsec. 2.1).

Thus, the problem of finding a solution to the NLWEE is reduced to finding an appropriate transformation that leads to eq. (4), which, in this sense, is the basis of the following analysis (for reference purposes it is called the "basic equation" of the associated NLWEE).

As is well known [7,8] the solutions $f(z)$ of (4) can be expressed in terms of Weierstrass' elliptic function $\wp$. It reads

$$f(z) = f_0 + \frac{R'(f_0)}{4[\wp(z; g_2, g_3) - \frac{1}{24}R''(f_0)]}, \tag{5}$$

where the primes denote differentiation with respect to f and f_0 is a simple root of $R(f)$.

The invariants g_2, g_3 of Weierstrass' elliptic function $\wp(z; g_2, g_3)$ are related to the coefficients of $R(f)$ by [9]

$$g_2 = \alpha\epsilon - 4\beta\delta + 3\gamma^2 \quad , \tag{6}$$

$$g_3 = \alpha\gamma\epsilon + 2\beta\gamma\delta - \alpha\delta^2 - \gamma^3 - \epsilon\beta^2 \quad . \tag{7}$$

The invariants and the discriminant (of $\wp$ and R [9])

$$\triangle = g_2^3 - 27g_3^2, \tag{8}$$

are suitable to classify the behaviour of $f(z)$. The conditions [11]

$$\triangle \neq 0 \quad \text{or} \quad \triangle = 0 \quad , \quad g_2 > 0, \quad g_3 > 0. \tag{9}$$

lead to periodic solutions, whereas the conditions

$$\triangle = 0, \quad g_2 \geq 0 \quad , \quad g_3 \leq 0 \tag{10}$$

are associated with solitary solutions.

If $\triangle = 0$, the solution (5) can be simplified because $\wp(z; g_2, g_3)$ degenerates into trigonometric or hyperbolic functions [13, pp. 651–652].

Thus, in this case, periodic solutions according to eq. (5) are determined by

$$f(z) = f_0 + \frac{R'(f_0)}{4\left[-\frac{e_1}{2} - \frac{R''(f_0)}{24} + \frac{3}{2}e_1 \csc^2(\sqrt{\frac{3}{2}e_1}z)\right]}, \quad \triangle = 0,\ g_3 > 0, \tag{11}$$

and solitary wave like solutions by

$$f(z) = f_0 + \frac{R'(f_0)}{4\left[e_1 - \frac{R''(f_0)}{24} + 3e_1 \mathrm{csch}^2(\sqrt{3e_1}z)\right]}, \quad \triangle = 0,\ g_3 < 0, \tag{12}$$

$$f(z) = f_0 + \frac{6R'(f_0)z^2}{24 - R''(f_0)z^2}, \quad \triangle = 0,\ g_3 = 0, \quad R''(f_0) < 0, \tag{13}$$

where $e_1 = \sqrt[3]{|g_3|}$ in eq. (11) and $e_1 = \frac{1}{2}\sqrt[3]{|g_3|}$ in eq. (12).

In general, $f(z)$ (according to eq. (5)) is neither real nor bounded. Conditions for real and bounded solutions $f(z)$ can be obtained by considering the phase diagram of $R(f)$ [12, p. 15]. They are denoted as "phase diagram conditions" (PDC) in the following. Examples of a phase diagram analysis is given in refs. [10,11,12].

2.1 Balancing principle

As mentioned above, the exponent M in eq. (3) is determined by use of a balancing principle. In this subsection a balancing principle for the approach outlined above is evaluated.

As a consequence of the transformation $u \to g(f(z))$ (cf. eq. (3)), the NLWEE can be rewritten in terms of $g(f)$ and its derivatives. To determine the leading M exponent

$$\tilde{g}(f) = a_0 + a_M f^M \tag{14}$$

has to be substituted into the NLWEE in question that leads to an expression of the form

$$\widetilde{P}(f) = 0, \tag{15}$$

$$\sqrt{\alpha f^4+4\beta f^3+6\gamma f^2+4\delta f+\epsilon}\ \widetilde{Q}(f)=0, \tag{16}$$

$$\widetilde{P}(f)+\sqrt{\alpha f^4+4\beta f^3+6\gamma f^2+4\delta f+\epsilon}\ \widetilde{Q}(f)=0, \tag{17}$$

where $\widetilde{P}$ and $\widetilde{Q}$ are polynomials in f. To determine M in ansatz (3) - according to type (15), (16), (17) - the polynomial equations $\widetilde{P}(f)=0$, $\widetilde{Q}(f)=0$ or the equation

$$\widetilde{P}(f)^2-R(f)\,\widetilde{Q}(f)^2=0 \tag{18}$$

is considered and every two possible highest exponents are equated. To avoid applying this procedure for every NLWEE anew, the highest possible exponents of the polynomials that occur for special derivatives in the nonlinear wave and evolution equation in question have been evaluated. One obtains from equation (4)

$$\frac{d}{dz}\rightarrow\sqrt{\alpha f^4+4\beta f^3+6\gamma f^2+4\delta f+\epsilon}\frac{d}{df}, \tag{19}$$

$$\frac{d^2}{dz^2}\rightarrow(2\alpha f^3+6\beta f^2+6\gamma f+4\delta)\frac{d}{df}, \tag{20}$$

$$\begin{aligned}\frac{d^3}{dz^3}\rightarrow&(6\alpha f^2+12\beta f+6\gamma)\sqrt{\alpha f^4+4\beta f^3+6\gamma f^2+4\delta f+\epsilon}\frac{d}{df}\\&+(2\alpha f^3+6\beta f^2+6\gamma f+4\delta)\sqrt{\alpha f^4+4\beta f^3+6\gamma f^2+4\delta f+\epsilon}\frac{d^2}{df^2}.\end{aligned} \tag{21}$$

Consideration of these relations and eq. (14) leads to the following list of the highest possible exponents (HPEs) (cf. tab. 1).

Function	g	g_z	g_{zz}	g_{zzz}	$g^n\, g_z$	...
HPE of $\widetilde{P}$	M	0	$M+2$	0	0	...
HPE of $\widetilde{Q}$	0	$M-1$	0	$M+1$	M (n + 1) - 1	...

Table 1
Highest possible exponent (HPE) of function $\widetilde{g}$ and its derivatives according to (14) and equations (19) - (21).

2.2 Solutions to the BBM equation

Using a transformation [1]

[1] Balancing g_{zzz} and $g^n\, g_z$ yields $M=\frac{2}{n}$ according to tab. 1. To simplify the comparison with the ansaetze in [4] $M=\frac{1}{n}$ is chosen. It is obvious, that solutions

$$
\begin{aligned}
u(x,t) &= g(z),\ z = \mu(x-ct), \\
g(z) &= h(z)^{\frac{1}{n}}, \\
h(z) &= \sum_{j=0}^{2} a_j f^j,
\end{aligned} \tag{22}
$$

where f is supposed to obey eq. (4), leads to a polynomial equation $P(f) = 0$. Setting the coefficients equal to zero yields

$$
\begin{aligned}
\alpha &= -\frac{a a_2 n^2}{4(n+4)(n+1)\mu^2},\ \beta = -\frac{a a_1 n^2}{8(n+4)(n+1)\mu^2}, \\
\gamma &= \frac{n^2(-3a(a_1^2+4a_0a_2)+a_2(c-1)(n+4)(n+1))}{96a_2(n+4)(n+1)}, \\
\delta &= \frac{a_1 n^2(a(a_1^2-12a_0a_2)+a_2(c-1)(n+4)(n+1))}{64a_2^2(n+1)(n+4)\mu^2}, \\
\epsilon &= -\frac{n^2(a(a_1^4-12a_0a_1^2a_2+48a_0^2a_2^2)+a_2(a_1^2-8a_0a_2)(c-1)(n+4)(n+1))}{64a_2^3(n+1)(n+4)\mu^2}, \\
a_0 &= \frac{a_1^2}{4a_2} \vee a_0 = \frac{a a_1^2 + a_2(c-1)(n+4)(n+1)}{4 a a_2}.
\end{aligned} \tag{23}
$$

According to eqs. (6) - (8) the invariants of the Weierstrass' elliptic function and the discriminant read

$$
g_2 = \frac{(c-1)^2 n^4}{3072\mu^4}, g_3 = -\frac{(c-1)^3 n^6}{884736\mu^6}, \triangle = 0 \text{ if } a_0 = \frac{a_1^2}{4a_2}; \tag{24}
$$

$$
\begin{aligned}
g_2 &= \frac{(c-1)^2 n^4}{192\mu^4}, g_3 = -\frac{(c-1)^3 n^6}{13824\mu^6}, \triangle = 0 \\
\text{if } a_0 &= \frac{a a_1^2 + a_2(c-1)(n+4)(n+1)}{4 a a_2}.
\end{aligned} \tag{25}
$$

As can be seen from eqs. (24) and (25), periodic solutions are given if $c < 1$, whereas $c > 1$ leads to solitary solutions, according to conditions (9), (10).

If $a_0 = \frac{a_1^2}{4a_2}$, periodic and solitary solutions can be evaluated according to eqs.

(26), (27) and (28) can easily be rewritten as $\widetilde{h}(z)^{\frac{2}{n}}$ as expected from the balancing result.

(22), (11) and (12) [2], respectively. The periodic solution reads

$$u(x,t) = \left\{ \frac{(c-1)(n+4)(n+1)\sec^2[\frac{n}{4}(x-ct)\sqrt{(1-c)}]}{4a} \right\}^{\frac{1}{n}}. \tag{26}$$

Solitary solutions are given by

$$u(x,t) = \left\{ \frac{(c-1)(n+4)(n+1)\mathrm{sech}^2[\frac{n}{4}(x-ct)\sqrt{(c-1)}]}{4a} \right\}^{\frac{1}{n}}, \tag{27}$$

$$u(x,t) = \left(\frac{\{aa_1^3 + 2a_2\sqrt{2aa_1^2(c-1)(n+2)(n+1)}\mathrm{sech}[\frac{n}{2}(x-ct)\sqrt{(c-1)}]\}^2}{16a^2a_1^4a_2} \right)^{\frac{1}{n}}. \tag{28}$$

If $a_0 = \frac{aa_1^2 + a_2(c-1)(n+4)(n+1)}{4aa_2}$ the basic equation has two double roots. Therefore, the general solution to eq. (4) [7,8]

$$f(z) = f_0 + \frac{\sqrt{R(f_0)}\frac{d\wp(z;g_2,g_3)}{dz} + \frac{1}{2}R'(f_0)[\wp(z;g_2,g_3) - \frac{1}{24}R''(f_0)] + \frac{1}{24}R(f_0)R'''(f_0)}{2[\wp(z;g_2,g_3) - \frac{1}{24}R''(f_0)]^2 - \frac{1}{48}R(f_0)R''''(f_0)}, \tag{29}$$

where f_0 is any constant, not necessarily a simple root of $R(f)$, has to be considered. Evaluation yields exactly solutions (26) and (27).

Equations (26), (27) and (28) present periodic and solitary solutions to the BBM equation. Because of the generalized ansatz there are no restrictions for a_1 and a_2. This may be advantageous to adapt the theoretical results to a physical problem that is modelled by the BBM equation.

3 Relationship between the approach above to some other ad-hoc methods

Recently, Wazwaz has introduced five ansaetze to solve the BBM equation [4]. Comparing these ansaetze with the approach outlined above, shows, that four of the ansaetze occur as special cases of the approach here. The sinh − cosh-ansatz III can only be classified as a special case if $n = 1$. It is remarkable, that this classification as "special cases" does not depend on the nonlinear

[2] The case $g_2 = g_3 = 0$ is neglected here, because according to the PDC there exist no bounded solutions.

equation in question, but is a characteristic of the approach outlined above. Therefore, features like the balancing principle, conditions for periodic and solitary wave solutions and the PDC can also be applied to the ansaetze given in [4]. This may be of interest because these ansaetze have also been applied succesfully to the Boussinesq equation [14].

As mentioned above the exponent M in eq. (3) can always be obtained by balancing[3]. Therefore, the base of the ansaetze in [4] has to be compared with eq. (4). The comparison is as follows: An ansatz $f(z)$ according to [4] with $z = x - ct$ is considered. The function f is differentiated with respect to z and $\left(\frac{df(z)}{dz}\right)^2$ is rewritten in terms of f. Comparing this expression with eq. (4) leads to the relationship between the parameters in the ansatz and the parameters in the basic equation. The results are presented in tab. 2.

It should be pointed out that the following classification of the ansaetze as subcases of a generalized ansatz (3), (4) has several advantages: Firstly, it is shown that ad-hoc methods can be unified to a certain extent. Secondly, periodic and solitary solutions can be deduced systematically according to conditions (9) and (10). Thirdly, certain procedures, e. g. balancing, has only to be done once and not for every ansatz anew. Finally, some of the remarks concerning solutions evaluated in ref. [4] (confered to as "W") can be approved from another point of view: The remark, that solutions given by eqs. (W20), (W21) are consistent with solutions (W18), (W19) can be approved by considering tab. 2. For $\lambda = \pm 1$ in the cosh I -ansatz and $\lambda = \pm i$ in the sinh I-ansatz the parameters of the basic equation are equal ($\alpha = 0$ in both cases), so that the same solutions have to be expected.

4 Summary

A method is proposed to obtain exact elliptic solutions to NLWEEs. The method is a generalization of some recently established ad-hoc methods [4] (cf. tab. 2) and, furthermore, includes conditions for periodic and solitary solutions. These are given compactly in terms of the invariants of the Weierstrass' elliptic function $\wp$. Periodic and solitary solutions can be deduced systematically from the general solution (5) and (29), respectively. Thus, a further promising ad-hoc method has been shown and, exemplary, solutions to the BBM equation have been evaluated. A balancing principle has been evaluated that simplifies the application of this method (cf. tab. 1). It seems that this

[3] In ref. [4] the exponent in the sinh − cosh-ansaetze I-III is directly given as $\frac{1}{n}$. This result is obtained by balancing here.

Name	Ansatz $f(z)$	Parameters of the basic equation	Comment
cosh I	$\frac{b}{1+\lambda\cosh(\mu\, z)}$	$\alpha=\frac{\mu^2}{b^2}(1-\lambda^2), \beta=-\frac{\mu^2}{2b}, \gamma=\frac{\mu^2}{6}, \delta=\epsilon=0$	special case of eq. (4)
sinh I	$\frac{b}{1+\lambda\sinh(\mu\, z)}$	$\alpha=\frac{\mu^2}{b^2}(1+\lambda^2), \beta=-\frac{\mu^2}{2b}, \gamma=\frac{\mu^2}{6}, \delta=\epsilon=0$	special case of eq. (4)
cosh II	$\frac{b}{1+\lambda\cosh^2(\mu\, z)}$	$\alpha=\frac{4\mu^2}{b^2}(1+\lambda), \beta=-\frac{\mu^2}{b}(2+\lambda), \gamma=\frac{2\mu^2}{3}, \delta=\epsilon=0$	special case of eq. (4)
sinh II	$\frac{b}{1+\lambda\sinh^2(\mu\, z)}$	$\alpha=\frac{4\mu^2}{b^2}(1-\lambda), \beta=-\frac{\mu^2}{b}(2-\lambda), \gamma=\frac{2\mu^2}{3}, \delta=\epsilon=0$	special case of eq. (4)
cosh III $n=1$	$\frac{b\cosh^2(\mu\, z)}{1+\lambda\cosh^2(\mu\, z)}$	$\alpha=\frac{4\lambda^2\mu^2(1+\lambda)}{b^2}, \beta=-\frac{\mu^2(3b\lambda^2+2b\lambda)}{b^2}$ $\gamma=\frac{2\mu^2(3b^2\lambda+b^2)}{3b^2}, \delta=-b\mu^2, \epsilon=0$	special case of eq. (4)
sinh III $n=1$	$\frac{b\sinh^2(\mu\, z)}{1+\lambda\sinh^2(\mu\, z)}$	$\alpha=\frac{4\lambda^2\mu^2(1-\lambda)}{b^2}, \beta=-\frac{\mu^2(-3b\lambda^2+2b\lambda)}{b^2}$ $\gamma=\frac{2\mu^2(-3b^2\lambda+b^2)}{3b^2}, \delta=b\mu^2, \epsilon=0$	special case of eq. (4)
cosine	$\lambda\cos(\mu z)$	$\alpha=\beta=\delta=0, \gamma=-\frac{\mu^2}{6}, \epsilon=\lambda^2\mu^2$	special case of eq. (4)
sine	$\lambda\sin(\mu z)$	$\alpha=\beta=\delta=0, \gamma=-\frac{\mu^2}{6}, \epsilon=\lambda^2\mu^2$	special case of eq. (4)
tanh-sech	$\tanh(\mu z)$	$\alpha=\mu^2, \beta=\delta=0, \gamma=-\frac{\mu^2}{3}, \epsilon=\mu^2$	special case of eq. (4)

Table 2: Relationship between the approach outlined here and some ad-hoc methods given in [4].

method is a useful and easily manageable tool with interesting features (balancing principle, conditions for periodic and solitary wave solutions, PDC) to find out exact solutions to nonlinear differential equations, especially solitary ones. It may be advantageous that this quite general method can lead to free parameters in the solution, as shown for the BBM equation.
It has recently been shown [5] for the Kadomtsev-Petviashvili equation and the nonlinear Schrödinger equation that if $\triangle \neq 0$ and special conditions hold, eq. (5) can be rewritten in terms of Jacobian elliptic functions and used as a start solution for a linear superposition principle that enlarges the solution set of periodic solutions [15]. For the Novikov-Veselov equation it has been shown previously [16], that a 2-solitary wave solution can be evaluated by linear superposition of two 1-solitary wave solutions to the Korteweg-de Vries equation. These 1-solitary wave solutions have been evaluated by the approach outlined above. Thus, this approach can also be used as a basis to construct periodic superposition solutions and multi-solitary wave solutions.

Acknowledgements

I would like to thank Prof. Wazwaz for sending me preprints of his interesting articles [4,14]. Furthermore, I would like to thank Prof. H. W. Schürmann for helpful discussions. The work was supported by the German Science Foundation (DFG) (Graduate College 695 "Nonlinearities of optical materials").

References

[1] H. Zhang, G. M. Wei and Y. T. Gao, On the general form of the Benjamin-Bona-Mahony equation in fluid mechanics, *Czechoslovak Journal of Physics* **51** (2001) 373–377.

[2] B. Wang and W. Yang, Finite-dimensional behaviour for the Benjamin-Bona-Mahony equation, *J. Phys. A: Math. Gen.* **30** (1997) 4877–4885.

[3] B. Wang,Regularity of attractors for the Benjamin-Bona-Mahony equation, *J. Phys. A: Math. Gen.* **31** (1998) 7635–7645.

[4] A. M. Wazwaz, New travelling wave solutions of different physical structures to generalized BBM equation, Phys. Lett. A **355** (2006) 358–362.

[5] H. W. Schürmann, V. S. Serov and J. Nickel, Superposition in nonlinear wave and evolution equations, *Int. J. Theor. Phys.* **45** (2006) 1057–1073.

[6] J. Nickel, V. S. Serov and H. W. Schürmann, Some elliptic traveling wave solutions to the Novikov-Veselov equation, *PIER* **61** (2006) 323–331.

[7] K. Weierstrass, *Mathematische Werke V* (Johnson, New York, 1915).

[8] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, 1927).

[9] K. Chandrasekharan, *Elliptic Functions* (Springer, Berlin, 1985).

[10] J. Nickel,Travelling wave solutions to the Kuramoto-Sivashinsky equation, *Chaos, Solitons and Fractals* **published online 20 March 2006**.

[11] H. W. Schürmann, Traveling-wave solutions of the cubic-quintic nonlinear Schrödinger equation, *Phys. Rev. E* **54** (1996) 4312–4320.

[12] P. G. Drazin, *Solitons* (Cambridge University Press, Cambridge, 1983).

[13] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* 9th ed. (Dover, New York, 1972).

[14] A. M. Wazwaz, New hyperbolic schemes for reliable treatment of Boussinesq equation, *Phys. Lett. A* **in press**.

[15] F. Cooper, A. Khare and U. Sukhatme, Periodic solutions of nonlinear equations obtained by linear superposition, *J. Phys. A: Math. Gen.* **35** (2002) 10085–10100.

[16] J. Nickel and H. W. Schürmann, 2-Soliton-solution of the Novikov-Veselov equation, *Int. J. Theor. Phys.* **published online 16 May 2006**.